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## The physics of white dwarfs

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**Abstract.** White dwarfs are the final remnants of low- and intermediate-mass stars. Their evolution is essentially a cooling process that lasts for  $\sim 10$  Ga ( $1 \text{ Ga} = 1 \times 10^9$  years) and allows one to obtain information about the age of the Galaxy as well as about the past stellar formation rate in the solar neighbourhood. Therefore, it is important to identify all of the relevant sources of energy as well as the mechanisms that control its flow to the space. We show in this paper that the inclusion of a detailed treatment of phase transitions in Coulomb plasmas made up of a mixture of different chemical species is crucial, since their redistribution can keep the white dwarf warm for 0.5 to 9 Ga depending on the chemical composition and physical assumptions adopted.

### 1. Introduction

White dwarfs represent the last evolutionary stage of stars with masses smaller than  $10 \pm 2 M_{\odot}$ , with the upper mass limit not yet well known. Most of them are composed of carbon and oxygen, but white dwarfs with masses smaller than  $0.4 M_{\odot}$  are made of helium, while those more massive than  $\sim 1.05 M_{\odot}$  are made of oxygen and neon. The exact composition of the carbon–oxygen cores depends critically on the evolution during the previous asymptotic giant branch phase, and more specifically on the competition between the  $^{12}\text{C}(\alpha, \gamma)^{16}\text{O}$  reaction and the triple- $\alpha$  reaction, on the details of the stellar evolutionary codes and on the choice of several other nuclear cross sections. In a typical case, a white dwarf of mass  $0.58 M_{\odot}$ , the total amount of oxygen represents 62% of the total mass while its concentration in the central layers of the white dwarf can be as high as 85%.

In all cases, the core is surrounded by a thin layer of pure helium with a mass in the range  $10^{-2}$  to  $10^{-4} M_{\odot}$ . This layer is, in turn, surrounded by an even thinner layer of hydrogen with mass lying in the range  $10^{-4}$  to  $10^{-15} M_{\odot}$ . This layer is missing in 25% of cases. From the phenomenological point of view, white dwarfs containing hydrogen are classified as DA ones while the remaining ones (the non-DA ones) are classified as DO, DB, DQ, DZ and DC, depending on their spectral features, and constitute a sequence with decreasing temperature. The origin of these spectral differences and the relationship among them has not yet been elucidated although it is related to the initial conditions imposed by the evolution of AGB stars, the diffusion induced by gravity, thermal diffusion, radiative

levitation, convection at the H–He and He–core interfaces, proton burning, stellar winds and mass accretion from the interstellar medium.

The structure of white dwarfs is sustained by the pressure of degenerate electrons and these stars cannot obtain energy from thermonuclear reactions. Therefore, their evolution can be described just as a simple cooling process (Mestel 1952) in which the internal degenerate core acts as a reservoir of energy and the outer non-degenerate layers control the energy outflow. If it is assumed that the core is isothermal, which is justified by the high conductivity of the degenerate electrons, and that the envelope is thin, then

$$L \approx -\frac{dU_{\text{th}}}{dt} = -\bar{c}_V M_{\text{WD}} \frac{dT_c}{dt} \quad (1)$$

$$\frac{L}{M_{\text{WD}}} = f(T_c) \quad (2)$$

where  $U_{\text{th}}$  is the thermal content,  $\bar{c}_V$  is the average specific heat,  $T_c$  is the temperature of the (isothermal) core and  $f(T_c)$  is a function governed by the structure of the envelope that relates the temperature of the core to the luminosity of the star. All of the remaining symbols have their usual meaning. A simple calculation indicates that the lifetimes of these stars are very long,  $\sim 10$  Ga ( $1 \text{ Ga} = 1 \times 10^9$  years), and thus they retain important information about the past history of the Galaxy. In particular, it is possible to obtain the stellar formation rate and the ages of the different Galactic components: disc, halo and globular clusters.

The structural simplicity of white dwarfs also makes them ideal laboratories for testing new physics since any perturbation of the cooling rate immediately translates into a perturbation of the luminosity function or of the secular variation of the period of pulsation of variable white dwarfs. For instance, Isern *et al* (1992) used ZZ Ceti stars (a class of pulsating white dwarfs) to constrain the mass of the axions, whereas Blinnikov and Dunina-Barkovskaya (1994) used the properties of hot white dwarfs to constrain the magnetic momentum of the neutrino and García-Berro *et al* (1995) used the distribution of the coldest white dwarfs to bound any possible variation of the gravitational constant,  $G$ .

The influence of Coulomb interactions on the specific heat was recognized early by Kirshnitz (1960), Abrikosov (1960) and Salpeter (1961). This idea led Van Horn (1968) and Lamb and Van Horn (1975), in a natural way, to consider the influence of crystallization, and the subsequent release of latent heat, on white dwarf evolution. Later on, Stevenson (1980) and Mochkovitch (1983) examined the release of gravitational energy associated with the change of the chemical composition induced by crystallization in C–O mixtures. Finally, Isern *et al* (1991) examined the consequences of the deposition of  $^{22}\text{Ne}$ , the most abundant of the impurities present in a white dwarf, in the central regions, and Xu and Van Horn (1992) did the same calculation for  $^{56}\text{Fe}$ , the impurity second in importance.

It is clear that not only can white dwarfs provide information about the Galaxy, but also they can be considered excellent laboratories in which to study the behaviour of matter under extreme conditions. Furthermore, the simplicity of their structure and evolution makes them extremely useful as astroparticle physics laboratories. But to use them as such, it is necessary to have good observational background, good evolutionary models and good input physics. In this paper we will deal with the relevance of phase transitions in the cooling process of white dwarfs.

## 2. Overview of white dwarf evolution

The evolution of a white dwarf from the planetary nebula phase to its disappearance depends on the properties of the envelope and the core, and has been discussed in detail by Iben

and Tutukov (1984), Koester and Schönberner (1986), D'Antona and Mazzitelli (1989) and Isern *et al* (1997). To summarize, the cooling process can be roughly divided into four stages: neutrino cooling, fluid cooling, crystallization and Debye cooling.

(i) *Neutrino cooling*:  $\log(L/L_\odot) > -1.5$ . This stage is very complicated because of the dependence on the initial conditions of the star as well as on the complex and not yet well understood behaviour of the envelope. For instance, Iben and Tutukov (1984) found that the luminosity due to hydrogen burning through the pp chains would never stop and could become dominant at low luminosities,  $-3.5 \leq \log(L/L_\odot) \leq -1.5$ . It is worth noting that, if this were the case, the cooling rate would be similar to the normal one (i.e., the one obtained neglecting this source) and it would be observationally impossible to distinguish between the two possibilities. However, the importance of such a source strongly depends on the mass,  $M_H$ , of the hydrogen layer. If  $M_H \leq 10^{-4} M_\odot$ , the pp contribution quickly drops and never becomes dominant. Since asteroseismological observations seem to constrain the size of  $M_H$  well below this critical value, this source can be neglected. Fortunately, when neutrino emission becomes dominant, the different thermal structures converge to a unique one, guaranteeing the uniformity of the models with  $\log(L/L_\odot) \lesssim -1.5$ . Furthermore, since the time necessary to reach this value is  $\lesssim 8 \times 10^7$  years (D'Antona and Mazzitelli 1989) for any model, its influence on the total cooling time is negligible.

(ii) *Fluid cooling*:  $-1.5 \geq \log(L/L_\odot) \geq -3$ . The main source of energy is the gravothermal one. Since the plasma is not very strongly coupled ( $\Gamma < 179$ ), its properties are reasonably well known (Ségretain *et al* 1994). Furthermore, the flux of energy through the envelope is controlled by a thick non-degenerate layer with an opacity dominated by hydrogen (if present) and helium, and weakly dependent on the metal content. The main source of uncertainty is related to the chemical structure of the interior, which depends on the rate adopted for the  $^{12}\text{C}(\alpha, \gamma)^{16}\text{O}$  reaction and on the treatment given to semiconvection and overshooting. If this rate is high, the oxygen abundance is higher in the centre than in the outer layers, resulting thus in a reduction of the specific heat in the central layers where the oxygen abundance can reach values as high as  $X_O = 0.85$  (Salaris *et al* 1997).

(iii) *Crystallization*:  $\log(L/L_\odot) < -3$ . Crystallization introduces two new sources of energy: latent heat and sedimentation. In the case of Coulomb plasmas, the latent heat is small, of the order of  $k_B T_s$  per nucleus, where  $k_B$  is the Boltzmann constant and  $T_s$  is the temperature of solidification. Its contribution to the total luminosity is small,  $\sim 5\%$ , but not negligible (Shaviv and Kovetz 1976).

During the crystallization process, the equilibrium chemical compositions of the solid and liquid plasmas are not equal. Therefore, if the resulting solid flakes are denser than the liquid mixture, they sink towards the central region. If they are lighter, they rise upwards and melt when the solidification temperature, which depends on the density, becomes equal to that of the isothermal core. The net effect is a migration of the heavier elements towards the central regions with the subsequent release of gravitational energy (Mochkovitch 1983). Of course, the efficiency of the process depends on the detailed chemical composition and on the initial chemical profile, and it is maximum for a mixture made of half oxygen and half carbon uniformly distributed throughout the star.

(iv) *Debye cooling*. When almost all of the star has solidified, the specific heat follows Debye's law. However, the outer layers still have very large temperatures as compared with the Debye one, and since their total heat capacity is still large enough, they prevent the sudden disappearance of the white dwarf in the case, at least, of thick envelopes (D'Antona and Mazzitelli 1989).

### 3. Physics of crystallization

#### 3.1. The carbon–oxygen mixture

The first calculation of a phase diagram for C/O mixtures was done by Stevenson (1980) who obtained a eutectic shape. This result was a consequence of his assumption that the solid was entirely random. Under this assumption the free energy was given by  $F \sim -0.9\Gamma$ , with  $\Gamma = \bar{Z}^{5/3}\Gamma_e$ , where  $\bar{Z} = \sum Z_i x_i$  and  $x_i$  is the abundance by number. Since the mixture retains some short-range order, the free energy is then given by the linear mixing rule,  $F_{\text{lm}} \sim -0.9\bar{Z}^{5/3}\Gamma_e$ , as was proved later on. Therefore the solid phase was less stable, and thus  $F > F_{\text{lm}}$ , consequently resulting in a eutectic behaviour of the phase diagram.

Barrat *et al* (1988) used the density functional theory of freezing and the mean-spherical approximation, but with the same diameter for the two chemical species, to compute the correlation between the particles. As a consequence they obtained a phase diagram of the spindle form. Moreover, they used the value  $\Gamma_C = 168$  for the crystallization of the pure phase. Ichimaru *et al* (1988) performed a similar calculation in the framework of density functional theory but using the improved hypernetted-chain approximation to compute the correlation functions. They found an azeotropic phase diagram with an azeotropic concentration of carbon  $x_C = 0.16$ . Finally, Ségretain and Chabrier (1993) extended the calculations of Barrat *et al* (1988), considering the effects of the different diameters of the two chemical species, and essentially found the same results as Barrat *et al* (1988). It is interesting to notice here that for the astrophysical applications and the usual abundances of carbon and oxygen found in white dwarf interiors, the three phase diagrams lead to the same results when all of them use the same value  $\Gamma_C = 180$  for the Coulomb coupling parameter at crystallization.

Because of the spindle shape of the phase diagram of C/O mixtures, the solid formed upon crystallization is richer in oxygen than the liquid and therefore denser. The density excess can be estimated using the condition of pressure continuity:

$$\frac{\delta\rho}{\rho} \simeq -\frac{\delta P_i}{\gamma P_e} - \frac{\delta Y_e}{Y_e} \quad (3)$$

where  $P_i$  and  $P_e$  are the ionic and electronic pressures respectively,  $\gamma$  is the electron adiabatic index and  $Y_e$  is the number of electrons per nucleon. For a  $0.6 M_\odot$  white dwarf with equal amounts of carbon and oxygen,  $\delta\rho/\rho \simeq 10^{-4}$ . Therefore the solid settles down at the core of the star and the lighter liquid left behind is redistributed by Rayleigh–Taylor instabilities (Stevenson 1980, Mochkovitch 1983). The result is an enrichment of oxygen in the central layers and its depletion in the outer ones.

The efficiency of convective mixing can be easily computed. Let  $v_{\text{crys}}$  be the velocity of the solidification front into the C/O mixture. The mass flux of carbon released by the front in the liquid phase is

$$F_C^{\text{crys}} = \rho v_{\text{crys}}(X_C^l - X_C^s) = \rho v_{\text{crys}} \Delta X_C \quad (4)$$

where  $X_C^l$  and  $X_C^s$  are the carbon mass fractions in the liquid and the solid respectively. This velocity, which can be obtained from the models, is very small,  $\sim 0.1$  cm per year.

The criterion for convective instability taking into account heat conduction from the convective eddies is (Stevenson and Salpeter 1977)

$$\chi > k\epsilon \quad (5)$$

with

$$\chi = -\frac{H_P}{\rho c_s^2} \mu \left( \frac{\partial P}{\partial \mu} \right)_{\rho, T} \left( \frac{1}{\mu} \frac{d\mu}{dr} \right) \quad (6)$$

$$\epsilon = \frac{H_P}{\rho c_s^2} T \left( \frac{\partial P}{\partial T} \right)_{\rho, \mu} \left[ \frac{1}{T} \frac{dT}{dr} - (\Gamma_3 - 1) \frac{1}{\rho} \frac{d\rho}{dr} \right] \quad (7)$$

$$k = \frac{\tau_{\text{cond}}}{\tau_{\text{cond}} + \tau_{\text{conv}}} \quad (8)$$

where  $H_P$  is the pressure scale height,  $c_s$  is the sound velocity,  $\mu$  is the mean molecular weight,  $\Gamma_3 - 1 = (\partial \log T / \partial \log \rho)_{\text{ad}}$ , and  $\tau_{\text{cond}}$  and  $\tau_{\text{conv}}$  are the conductive and the convective characteristic times, respectively. If  $l$  is the mixing length, the convective velocity can be written as

$$v_{\text{conv}} = c_s (\chi - k\epsilon)^{1/2} \frac{l}{H_P} \quad (9)$$

and the characteristic times are given by

$$\tau_{\text{conv}} = \frac{l}{v_{\text{conv}}} \quad (10)$$

$$\tau_{\text{cond}} = \frac{l^2}{K_T} \quad (11)$$

where  $K_T$  is the thermal conductivity. If the superadiabaticity is small (Isern *et al* 1997), the carbon mass flux released is efficiently carried away by convection and the gradient of the carbon mass fraction is given by

$$\left| \frac{dX_C}{dr} \right| \simeq 4 \times 10^{-12} \left( \frac{v_{\text{crys}}}{0.1 \text{ cm per year}} \right) \left( \frac{\Delta X_C}{0.1} \right) \left( \frac{100 \text{ cm}^2 \text{ s}^{-1}}{K_T} \right) \left( \frac{1}{Pe} \right) \quad (12)$$

where  $Pe$  is the Peclet number,  $Pe = \tau_{\text{cond}} / \tau_{\text{conv}}$ . The gradient is so small that  $X_C$  varies by less than 1% in the convective region.

An upper limit for the superadiabaticity can be obtained by assuming that, due to the interaction of convection with rotation, the Rossby number

$$Ro = \frac{v_{\text{conv}}}{\omega l} \quad (13)$$

is equal to unity, where  $\omega$  is the angular velocity of the white dwarf (Stevenson and Salpeter 1977). Then,

$$\chi - k\epsilon = \left( \frac{H_P \omega}{c_s} \right)^2 = 3 \times 10^{-7} \left( \frac{H_P}{10^9 \text{ cm}} \right)^2 \left( \frac{c_s}{3 \times 10^8 \text{ cm s}^{-1}} \right)^{-2} \left( \frac{\Pi}{10 \text{ h}} \right) \quad (14)$$

where  $\Pi$  is the rotation period. We therefore conclude that, even when rotation is considered, convection is indeed an efficient mechanism for redistributing the carbon-rich fluid out from the crystallization front and that the liquid phase can be considered well mixed (see Isern *et al* (1997) for more details).

With the hypothesis of perfect mixing, the composition profile can be easily computed. Consider a partially solidified white dwarf of total mass  $M_{\text{WD}}$  containing a total amount of oxygen  $M_{\text{O}}$ . Its structure can be divided into three parts: a solid core of mass  $M_s$  and oxygen mass fraction  $X_s(m)$ , a liquid mantle of mass  $\Delta M$  homogenized by convection, with oxygen abundance  $X$ , and an outer, unperturbed region, with the initial oxygen profile  $X_{\text{O}}(m)$ . Therefore, the total mass of oxygen can be written as

$$M_{\text{O}} = \int_0^{M_s} X_s dm + X(M_1 - M_s) + \int_{M_1}^{M_{\text{WD}}} X_{\text{O}} dm \quad (15)$$

where  $M_1 = M_s + \Delta M$

After deriving this expression with respect to the solid mass and introducing the two conditions  $X_s(M_s) = (1 + \alpha)X$  and  $X = X_O(M_s)$ , where  $\alpha$ , which depends on  $X$ , is the degree of enrichment produced during crystallization, we obtain

$$\alpha X + \frac{dX}{dM_s}(M_1 - M_s) = 0 \quad (16)$$

and, introducing  $q = M_s/M_{WD}$  and  $q_1 = M_L/M_{WD}$ ,

$$\frac{dX}{dq}[q_1(X) - q] + \alpha(X)X = 0. \quad (17)$$

Notice the singularity that occurs at  $q = 0$  since  $q_1(X) = 0$  and also notice that if the initial profile is flat,  $q_L(X) = 1$ . Integration of this equation provides the final oxygen profile after crystallization (Salaris *et al* 1997).

### 3.2. The role of minor species

As already mentioned, the  $^{22}\text{Ne}$  isotope is the most abundant impurity in white dwarfs. Its abundance is directly related to the initial abundances of CNO elements, which, after the H-burning phase, become  $^{14}\text{N}$ . This isotope, in turn, becomes  $^{22}\text{Ne}$  after the series of reactions  $^{14}\text{N}(\alpha, \gamma)^{18}\text{O}(\alpha, \gamma)^{22}\text{Ne}$ , during the He-burning phase. Because of its high neutron number and the high sensitivity of degenerate structures to the electron number profile,  $^{22}\text{Ne}$  can induce a large release of gravitational energy if, as a consequence of crystallization, it migrates towards the centre during crystallization (Isern *et al* 1991). For stars of solar metallicity, the typical abundances are 1–2%. A similar effect can be produced by the deposition of  $^{56}\text{Fe}$  at the centre (Xu and Van Horn 1992). Typical abundances are in this case 0.1%.

The physics of the deposition of the minor species is intricate since it depends on the behaviour of a multicomponent phase diagram which is not known. A first step consists in assuming that the C/O/Ne or C/O/Fe mixtures behave as an effective binary mixture composed from neon (or iron) and an average element, representative of the C/O mixture.

Ségreain and Chabrier (1993) computed the crystallization diagram for arbitrary ionic mixtures as a function of the charge ratio and found that the phase diagram was of the spindle form for  $0.72 \lesssim Z_1/Z_2 < 1$ , of the azeotropic form for  $0.58 \lesssim Z_1/Z_2 \lesssim 0.72$  and of the eutectic form for  $Z_1/Z_2 \lesssim 0.58$ . In the case of a C/O mixture made up half and half of each species, the resulting average element has such a charge that the corresponding C/O/Ne phase diagram shows an azeotropic behaviour with an azeotropic abundance of  $X_a = 0.16$ , which means that white dwarfs are in the neon-poor side of the phase diagram. As a consequence, the solid in equilibrium with the liquid has a smaller concentration of neon and, since is lighter than the surrounding liquid, it will rise and melt in lower-density regions, so the neon concentration in the liquid will increase more and more until it reaches the azeotropic composition. This process of ‘distillation’ will continue until all of the  $^{22}\text{Ne}$  is collected in a central sphere of mass  $M_{WD}X_0(\text{Ne})/X_a(\text{Ne})$ .

Ségreain (1996) computed a preliminary ternary phase diagram that displays the expected behaviour at the binary limit (the spindle form for the C/O mixture, the azeotropic form for the C/Ne mixture and the spindle form for the O/Ne mixture). For small concentrations of neon, and temperatures well above the azeotropic temperature, the crystallization diagram is not affected by the presence of neon. However, as the temperature approaches the azeotropic one, the resulting solid is lighter than the surrounding liquid and the distillation process starts as in the previous case. The main difference is that it starts in

the outer layers instead of the central layers and the effect of separation is therefore much smaller.

#### 4. The energetics of the crystallization process

The local energy budget of the white dwarf can be written as

$$\frac{dL_r}{dm} = -\epsilon_v - P \frac{dV}{dt} - \frac{dE}{dt} \quad (18)$$

where all of the symbols have their usual meaning. If the white dwarf is made up of two chemical species with atomic numbers  $Z_0$  and  $Z_1$ , mass numbers  $A_0$  and  $A_1$ , and abundances by mass  $X_0$  and  $X_1$ , respectively ( $X_0 + X_1 = 1$ ), where the suffix 0 refers to the heavier component, this equation can be written as

$$-\left(\frac{dL_r}{dm} + \epsilon_v\right) = C_V \frac{dT}{dt} + T \left(\frac{\partial P}{\partial T}\right)_{V, X_0} \frac{dV}{dt} - l_s \frac{dM_s}{dt} \delta(m - M_s) + \left(\frac{\partial E}{\partial X_0}\right)_{T, V} \frac{dX_0}{dt} \quad (19)$$

where  $l_s$  is the latent heat of crystallization and  $dM_s/dt$  is the rate at which the solid core grows; the delta function indicates that the latent heat is released at the solidification front. Notice that chemical differentiation contributes to the luminosity not only through the compressional work, which is negligible, but also through the change in the chemical abundances, which leads to the last term of this equation. Notice, as well, that the largest contribution to  $L_r$  due to the change in  $E$  exactly cancels out the  $P dV$  work for *any* evolutionary change (with or without a compositional change). This is, of course, a well known result (Mestel 1952, Kovetz and Shaviv 1976, Lamb and Van Horn 1975, D'Antona and Mazzitelli 1989) that can be related to the release of gravitational energy (Isern *et al* 1997).

Integrating over the whole star, we obtain

$$\begin{aligned} L + L_v = & - \int_0^{M_{\text{WD}}} C_V \frac{dT}{dt} dm - \int_0^{M_{\text{WD}}} T \left(\frac{\partial P}{\partial T}\right)_{V, X_0} \frac{dV}{dt} dm \\ & + l_s \frac{dM_s}{dt} - \int_0^{M_{\text{WD}}} \left(\frac{\partial E}{\partial X_0}\right)_{T, V} \frac{dX_0}{dt} dm. \end{aligned} \quad (20)$$

The first term of the equation is the well known contribution of the heat capacity of the star to the total luminosity (Mestel 1952). The second term represents the contribution to the luminosity due to the change of volume. It is in general small since only the thermal part of the electronic pressure, the ideal part of the ions and the Coulomb terms other than the Madelung term contribute (Kovetz and Shaviv 1976, Lamb and Van Horn 1975). However, when the white dwarf enters into the Debye regime, this term provides about 80% of the total luminosity preventing the sudden disappearance of the star (D'Antona and Mazzitelli 1989). The third term represents the contribution of the latent heat to the total luminosity at freezing. Since the latent heat of Coulomb plasmas is small, its contribution to the total luminosity is modest although not negligible. The fourth term represents the energy released by the chemical readjustment of the white dwarf, i.e. the release of the energy stored in the form of chemical potentials. This term is usually negligible in normal stars, since it is much smaller than the energy released by nuclear reactions, but it must be taken into account when all other energy sources are small.

The last term can be further expanded and written as (Isern *et al* 1997)

$$\int_0^{M_{\text{WD}}} \left(\frac{\partial E}{\partial X_0}\right)_{T, V} \frac{dX_0}{dt} dm = (X_0^{\text{sol}} - X_0^{\text{liq}}) \left[ \left(\frac{\partial E}{\partial X_0}\right)_{M_s} - \left\langle \frac{\partial E}{\partial X_0} \right\rangle \right] \frac{dM_s}{dt} \quad (21)$$



**Table 1.** The energy released by the chemical differentiation induced by crystallization and the corresponding delays.

Mixture	$\Delta E$ (erg)	$\Delta t$ (Ga)
C/O	$1.95 \times 10^{46}$	1.81
A/Ne	$1.52 \times 10^{47}$	9.09
A/Fe	$2.00 \times 10^{46}$	1.09
C/O/Ne	$0.20 \times 10^{46}$	0.60

where

$$\left\langle \frac{\partial E}{\partial X_0} \right\rangle = \frac{1}{\Delta M} \int_{\Delta M} \left( \frac{\partial E}{\partial X_0} \right)_{T,V} dm \quad (22)$$

and it is possible to define the total energy released per gram of crystallized matter:

$$\epsilon_g = -(X_0^{\text{sol}} - X_0^{\text{liq}}) \left[ \left( \frac{\partial E}{\partial X_0} \right)_{M_s} - \left\langle \frac{\partial E}{\partial X_0} \right\rangle \right]. \quad (23)$$

The square bracket is negative since  $\partial E/\partial X_0$  is negative and essentially depends on the density, which monotonically decreases outwards.

The delay introduced by solidification can be easily estimated to a good approximation if it is assumed that the luminosity of the white dwarf is just a function of the temperature of the nearly isothermal core (Isern *et al* 1997). In this case

$$\Delta t = \int_0^{M_{\text{WD}}} \frac{\epsilon_g(T_c)}{L(T_c)} dm \quad (24)$$

where  $\epsilon_g$  is the energy released per unit of crystallized mass and  $T_c$  is the temperature of the core when the crystallization front is located at  $m$ . Of course, the total delay essentially depends on the transparency of the envelope. Any change in one sense or another can amplify or damp the influence of solidification and for the moment there are no reliable envelope models for low luminosities.

Table 1 displays the energy released in an, otherwise typical,  $0.6 M_{\odot}$  white dwarf and the delays introduced by the different cases of solidification discussed here assuming that the envelope is the same as that considered by Isern *et al* (1997) and that the white dwarf is half carbon and half oxygen. The symbol A represents the effective binary mixture. Its use is probably justified in the case of impurities of very high atomic number such as iron. However, in the case of Ne this assumption is probably doubtful, as has been shown by Ségretain (1996).

## 5. The age of the Galactic disc

The white dwarf luminosity function is defined as the number of white dwarfs with bolometric magnitude  $M_{\text{bol}}$  per cubic parsec and unit bolometric magnitude. The first luminosity functions were obtained by Weidemann (1968), Kovetz and Shaviv (1976) and by Sion and Liebert (1977). Later on, Fleming *et al* (1986) substantially improved this function for the domain of white dwarfs brighter than  $\log(L/L_{\odot}) \approx -3.3$ , while Liebert *et al* (1988) did the same for the coolest ones. The two main properties of this empirical luminosity function are its monotonic increase when the luminosity decreases, which indicates the cooling nature of the evolution of white dwarfs, and the existence of a shortfall at  $\log(L/L_{\odot}) \approx -4.5$ , which is interpreted as a consequence of the finite age

of the Galaxy. From the comparison between this empirical function and the theoretical one, it is possible to obtain the age of the disc,  $T$ , and the star formation rate,  $\Psi(t)$ , as a function of time.

The luminosity function can be computed as

$$n(l) \propto \int_{M_i}^{M_s} \Phi(M) \Psi(T - t_{\text{cool}}(l, M) - t_{\text{MS}}(M)) \tau_{\text{cool}}(l, M) dM \quad (25)$$

where  $l$  is the logarithm of the luminosity in solar units,  $M$  is the mass of the parent star (for convenience all white dwarfs are labelled with the mass of the main-sequence progenitor),  $t_{\text{cool}}$  is the time taken to cool down to luminosity  $l$ ,  $\tau_{\text{cool}} = dt/dM_{\text{bol}}$  is the characteristic cooling time,  $M_s$  and  $M_i$  are the maximum and the minimum masses of the main-sequence stars able to produce a white dwarf of luminosity  $l$ ,  $t_{\text{MS}}$  is the main sequence lifetime of the progenitor of the white dwarf and  $T$  is the age of the population under study. The remaining quantities, the initial mass function,  $\Phi(M)$ , and the star formation rate,  $\Psi(t)$ , are not known *a priori*, and depend on the astronomical properties of the stellar population under study.

In order to compare with the observations properly, it is convenient to bin this function in intervals of magnitude  $\Delta M_{\text{bol}}$ , usually one or half magnitudes, in the following way:

$$\langle n(l) \rangle_{\Delta l} = \frac{1}{\Delta l} \int_{l-0.5\Delta l}^{l+0.5\Delta l} n(l) dl \quad (26)$$

where  $\Delta l$  is the size of the luminosity bin that corresponds to  $\Delta M_{\text{bol}}$ . If the cooling sequences are those from Salaris *et al* (1997), which do not include the effects of Ne and Fe deposition, the age of the disc obtained in this way is 9.2 Ga. Table 2 displays the uncertainties in the time necessary to fade to  $\log(L/L_{\odot}) = -4.5$ . In the lower section of this table the additive contributions to the uncertainty due to the physics of crystallization are shown, whereas the upper section describes the uncertainties due to the rest of the input physics. As can be seen, the major contribution is provided by the minor chemical species.

It is important to notice here that the binning procedure smooths and ultimately erases the spikes introduced by the sedimentation of Ne and Fe in the observational luminosity function (Hernanz *et al* 1994). Therefore, in order to detect the influence of these impurities observationally, it would be necessary to have high-resolution luminosity functions.

**Table 2.** Uncertainties in the estimates of the cooling times of white dwarfs.

Input	$\Delta t$ (Ga)	Comments
DA/non-DA	$\lesssim 3.0$	Very uncertain
Core composition	$\lesssim 0.5$	Depending on $^{12}\text{C}(\alpha, \gamma)^{16}\text{O}$
Opacity	$\lesssim 0.4$	
Metals in the envelope	$\approx 0.2$	
Additive contributions of the crystallization process		
C/O	0.8–1.2	Depending on $^{12}\text{C}(\alpha, \gamma)^{16}\text{O}$
Fe	$\lesssim 1.3$	
Ne	$\lesssim 9.0$	Binary mixture
	$\lesssim 0.5$	Ternary mixture
Observational	1–2	

## 6. Conclusions

The sedimentation of chemical species induced by crystallization is one of the major sources of energy of the coolest white dwarf stars. The delay introduced by the C/O partial separation is of the order of 1 Ga (this quantity depends on which model of the atmosphere is adopted). Minor species present in the white dwarf can also introduce huge delays that can range from 0.5 to 9 Ga. This uncertainty will be resolved when good ternary phase diagrams are available.

From the observational point of view, the obtaining of luminosity functions of the disc or globular clusters with resolutions in magnitude better than 0.5 magnitudes could easily allow testing of the different phase diagrams. Moreover, asteroseismology (Winget 1998) could also provide direct evidence as regards the crystallization process.

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